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LETTER TO THE EDITOR

On the resolution of ordering ambiguities associated with the path integral representation of stochastic processes

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Abstract. Simple path integral approaches to multiplicative stochastic processes are severely hampered by an inherent ambiguity in the underlying Lagrangian associated with operator ordering problems. We show how, given the Fokker-Planck equation, the ambiguity can be resolved. Direct consequences for noise-induced transitions, reaction-diffusion models and polymer dynamics are briefly discussed.

Path integrals offer a natural and powerful approach to the stochastic dynamics of complex systems. In particular, the multiplicative stochastic process underlying noise-induced transitions (Horsthemke and Lefever 1984), reaction-diffusion models (Elderfield 1985a, Elderfield and Vvedensky 1985a, b), polymer dynamics (Stepanow 1984) and kinetic growth models (Elderfield 1985b) have recently attracted special attention. Unhappily for multiplicative systems, the use of path integral representations is severely hampered by the existence of an inherent ambiguity in the choice of Lagrangian (De Witt 1957). Many authors (Leschke and Schmutz 1976, Langouche *et al* 1979) have shown that for path integrals, different temporal discretisations (or operator orderings) lead to superficially different Lagrangians $\mathscr{L}_\alpha$; $\alpha \in (0, 1)$ an arbitrary parameter. Consistency is demonstrated by observing that for all α important identities associated with normalisation and causality are satisfied; however, it is then generally assumed that a free choice of α is available (Langouche *et al* 1979). We argue here that such a point of view is erroneous (or at least gravely misleading) and moreover, given the Fokker-Planck equation, α is fully determined if the underlying stochastic dynamics is multiplicative. Particularly for noise-induced transitions the choice of α is crucial if the path integral is to describe the physics correctly. On the other hand, for additive stochastic processes, no constraint is placed on α by the present study.

We consider for simplicity the Fokker-Planck equation (normally ordered)

$$\frac{\partial}{\partial t}P(x, t) = \frac{\partial}{\partial x}\left(A(x)P(x, t) + \frac{\partial}{\partial x}(B(x)P(x, t))\right) \tag{1}$$

associated with the Ito stochastic equation

$$dx = -A(x)\,dt + (2B(x))^{1/2}\,dW \tag{2}$$

which by definition for $B'(x) \neq 0$ is multiplicative in character (Gardiner 1983). To derive the path integral representation we follow Langouche *et al* (1979) and Faddeev

(1975) and associate with (1) a quantum problem described by the Hamiltonian

$$\hat{H} = \hat{p}(A(\hat{x}) - \mathrm{i}\hat{p}B(\hat{x})) \tag{3}$$

where the operators $\hat{p}$, $\hat{x}$ satisfy the familiar commutation rules

$$[\hat{p}, \hat{x}] = \mathrm{i} \qquad [\hat{p}, \hat{p}] = 0 = [\hat{x}, \hat{x}]. \tag{4}$$

Using the conventional bra and ket notation ($\hat{x}|x\rangle = x|x\rangle$, $\hat{p}|p\rangle = p|p\rangle$, $\langle p|x\rangle = (1/2\pi)\exp(\mathrm{i}px)$) one finds that the Fokker-Planck distribution $P(x, t|x_0, t_0)$ satisfies

$$P(x, t|x_0, t_0) = \langle x|\hat{U}(t, t_0)|x_0\rangle \tag{5}$$

where $\hat{U}$ is an evolution operator satisfying

$$\begin{aligned} &\mathrm{i}\frac{\partial}{\partial t}\hat{U}(t, t_0) = \hat{H}(\hat{p}, \hat{x})\hat{U}(t, t_0) \\ &\hat{U}(t_0, t_0) = \hat{I}. \end{aligned} \tag{6}$$

Discretising the temporal interval (t_0, t), one is then led directly to the path integral representation via the familiar steps ($t_k = t_0 + k\varepsilon$, $k = 1, 2 \ldots N$, $t_{N+1} = t$)

$$\begin{aligned} P(x, t|x_0, t_0) &= \lim_{N\to\infty}\left(\int \prod_{i=1}^{N} \mathrm{d}x_i \prod_{j=1}^{N+1} \langle x_j|\hat{U}(t_j, t_{j-1})|x_{j-1}\rangle\right) \\ &= \lim_{N\to\infty}\left(\int \prod_{i=1}^{N} \mathrm{d}x_i \prod_{j=1}^{N+1} \mathrm{d}p_i \right. \\ &\quad \left.\times\exp\left[\!\left[-\mathrm{i}\varepsilon\left\{\sum_{k=1}^{N} p_x\left[\left(\frac{\alpha_x - \alpha_{k-1}}{\varepsilon}\right) + A(\alpha_{k-1}) - \mathrm{i}p_kB(\alpha_{k-1})\right]\right\}\right]\!\right]\right. \\ &\equiv \int [\mathrm{d}x(t)] \int [\mathrm{d}p(t)]\delta(x(t_0) - x_0)\delta(x(t) - x)\exp\left(-\mathrm{i}\int_{t_0}^{t} \mathrm{d}t\mathscr{L}\right) \end{aligned} \tag{7}$$

where the Lagrangian $\mathscr{L}$ is defined by

$$\mathscr{L} = p(t)(\dot{\alpha}(t) + A(\alpha(t)) - \mathrm{i}p(t)B(\alpha(t))). \tag{8}$$

Furthermore, introducing the Heisenberg operators $\hat{x}(t)$, $\hat{p}(t)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) \equiv \mathrm{i}[\hat{H}, \hat{x}(t)] = -A(\hat{x}(t)) + 2\mathrm{i}\hat{p}(t)B(\hat{x}(t)) \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}(t) \equiv \mathrm{i}[\hat{H}, \hat{p}(t)] = -\hat{p}(t)\frac{\partial}{\partial\hat{x}(t)}A(\hat{x}(t)) - \mathrm{i}\hat{p}^2(t)\frac{\partial}{\partial\hat{x}(t)}B(\hat{x}(t)) \tag{10}$$

one finds in an analogous fashion that

$$\langle x(t_1)x(t_2)\ldots x(t_k)\rangle = \int \mathrm{d}x\langle x|\hat{T}(\hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_k))|x_0\rangle \tag{11}$$

where $\hat{T}$ denotes the time ordering operator.

In this way one reproduces the path integrals described by Stepanow (1984) for the polymer problem, by Elderfield (1985a) for reaction-diffusion models and Elderfield (1985b) for kinetic growth models ($\alpha = 0$). To uncover the ambiguity we follow

Langouche *et al* (1979) and consider the superficially irrelevant modification

$$\hat{H}=(1-\alpha)(\hat{p}A(\hat{x})-\mathrm{i}\hat{p}^2B(\hat{x}))+\alpha(A(\hat{x})\hat{p}-\mathrm{i}\hat{B}(\hat{x})\hat{p}^2)+\alpha([\hat{p},A(\hat{x})]-\mathrm{i}[(\hat{p})^2\,B(\hat{x})]) \tag{12}$$

for $\alpha\in(0,1)$. Repeating the above computation, one now finds the Lagrangian $\mathscr{L}_\alpha$ given by

$$\mathscr{L}_\alpha=\mathscr{L}+\alpha\left(\mathrm{i}\frac{\partial A}{\partial x}+2p\frac{\partial B}{\partial x}\right). \tag{13}$$

Broadly, the difference between (8) and (13) can be attributed to the choice of $\hat{T}$ at equal time. Proceeding more cautiously in the chosen discretisation, one finds quite typically

$$\begin{aligned}\frac{\partial}{\partial l(t)}\langle x(t)\rangle&\equiv\langle p(t)x(t)\rangle\\ &=\lim_{\varepsilon\to0^+}\int \mathrm{d}x\langle x|\hat{T}(\hat{p}(t)((1-\alpha)\hat{x}(t-\tfrac12\varepsilon)+\alpha\hat{x}(t+\tfrac12\varepsilon))|x_0\rangle=-\mathrm{i}\alpha\end{aligned} \tag{14}$$

where we have used the property

$$\int \mathrm{d}x\langle x|\hat{p}(t)=\hat{0}. \tag{15}$$

Of course, as such response functions are naturally discontinuous at equal time (causality), one might reasonably hope that the α dependence induced by (14) cancels identically in all physical correlation/response functions. Performing a systematic expansion about the point x, $p=0$, one can test this hypothesis in great detail. For example, Langouche *et al* (1979) have shown that perturbatively one recovers for all α the identity

$$\langle p(t)\rangle=0 \tag{16}$$

associated with (15). Given the Fokker-Planck equation (1) or the underlying Langevin equation (2), there is, however, another important test. On the one hand, the path integral satisfies the identity

$$\int[\mathrm{d}x(t)]\int[\mathrm{d}p(t)]\frac{\partial}{\partial p(t)}\left[\exp\left(-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t\mathscr{L}_\alpha\right)\right]=0 \tag{17}$$

which leads directly to the equation of motion

$$\frac{\partial}{\partial t}\langle x(t)\rangle=-\langle A(x(t))\rangle-2\alpha\left\langle\frac{\partial B(x(t))}{\partial x(t)}\right\rangle \tag{18}$$

whilst, given the Fokker-Planck equation (1), one derives

$$\frac{\partial}{\partial t}\langle x\rangle(t)=-\langle A(x)\rangle(t) \tag{19}$$

using integration by parts. Comparing (17) and (18), we now see directly that for *multiplicative stochastic processes* ($B'(x)\neq0$) *only the choice* $\alpha=0$, *corresponding to a normally ordered Fokker-Planck equation, is consistent.* One cannot, therefore, freely choose α to simplify the diagrammatics as suggested by Langouche *et al* (1979). The implications of this for past work are happily minimal, for $\alpha = 0$ also gives the simplest diagrammatics (no tadpoles, see (14)). In direct contrast, for the description of noise-induced transitions (Horsthemke and Lefever 1984), this constraint is of crucial importance. Making on 'physical' grounds the symmetric (or unbiased) choice $\alpha = \frac{1}{2}$, one can erroneously relate such phenomena to the more familiar equilibrium transitions (Hohenberg and Halperin 1977, de Dominicis and Peliti 1978). Consider, for example, the 'genetic model' of Horsthemke and Lefever (1984)

$$\begin{aligned} A(x) &\equiv x - \tfrac{1}{2}\frac{\partial B}{\partial x} \\ B(x) &\equiv \sigma^2(1-x^2)^2 \end{aligned} \tag{20}$$

for which the stationary distribution is known (zero flux boundary conditions)

$$P_s(x) = \left(\frac{N}{B(x)}\right)^{1/2} \exp\left(-\int^x \mathrm{d}x \frac{x}{B(x)}\right) \qquad (N \equiv \text{normalisation constant}). \tag{21}$$

As the intensity σ^2 of the noise is increased, the distribution $P_s(x)$ develops from a Gaussian characteristic of the underlying damped harmonic oscillator to a bimodal form, i.e. the system undergoes a 'noise-induced transition'. By direct differentiation the extreme of (21) satisfy

$$x[1-\sigma^2(1-x^2)] = 0 \tag{22}$$

so one identifies the threshold as $\sigma_c^2 = 1$ for this transition. On the other hand, given the Lagrangian $\mathscr{L}_\alpha$ (13), a natural 'mean-field approximation' presents itself in the form of the extremum equations

$$0 = \frac{\partial \mathscr{L}_\alpha}{\partial p_c} = \left(\frac{\partial x_c}{\partial t} + A(x_c)\right) + 2\alpha \frac{\partial}{\partial x_c} B(x_c) \tag{23}$$

$$0 = \frac{\partial \mathscr{L}_\alpha}{\partial x_c} = \left(-\frac{\partial p_c}{\partial t} + p_c \frac{\partial}{\partial x_c} A(x_c) - \mathrm{i} p_c^2 \frac{\partial}{\partial x_c} B(x_c)\right) + \alpha\left(\mathrm{i}\frac{\partial^2}{\partial x_c^2} A(x_s) + 2p_c \frac{\partial^2}{\partial x_c^2} B(x_c)\right). \tag{24}$$

Comparing (23) with (22), we see that for $\alpha = \frac{1}{2}$ one can recover the 'noise-induced transition' of Horsthemke and Lefever (1984) ($p_c = 0$, see (16)). However, as shown above, *we must take* $\alpha = 0$, *so the apparent agreement is totally illusory.* Of course, to avoid violating (16), the mean-field relation (24) is to be interpreted as a loop subtraction (de Dominicis and Peliti 1978) in the normal manner.

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